Note on Weierstrass' Methods in the Theory of Elliptic Functions.

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The work of Professor Weierstrass in the modern function-theory is of such commanding importance that it may not be out of place to give a clear and elementary account of his somewhat peculiar nomenclature and methods for the benefit of those English readers who have not had the opportunity of listening to his lectures. This is especially desirable in the theory of doubly-periodic functions, where his symbols and methods differ not a little from those of Jacobi and his predecessors. The only connected and systematic statement of Weierstrass' methods in this field is contained in the "*Formeln und Lehrsätze zum Gebrauche der elliptischen Functionen. Nach Vorlesungen und Aufzeichnungen des Herrn K. Weierstrass bearbeitet und herausgegeben von H. A. Schwarz. Göttingen,* 1882." Professor Schwarz has prepared this little work with infinite pains for the use of his own and Weierstrass' students exclusively. I rely in the following chiefly on this work, on a large number of lithographed formulæ prepared by Prof. Schwarz, and on notes of lectures on this topic by Profs. Schwarz and Weierstrass.

We commence with the sigma-function, to designate which Weierstrass employs a slightly altered form of the Greek sigma. The simplest possible analytic function, which for all finite values of the argument retains the character of an integral function, and which becomes an infinitesimal of the first order for $u = 0$ and $u = w$, where $w = 2\mu\omega + 2\mu'\omega'$, is represented by the formula

$$\sigma u = u\Pi'_w\left(1 - \frac{u}{w}\right)e^{\frac{u}{w} + \frac{1}{2}\frac{u^2}{w^2}},$$

where 2ω, $2\omega'$ are the so-called "*periods,*" and μ, μ' assume all real integral values, positive and negative, excepting only the combination $\mu = 0$, $\mu' = 0$; which exception is indicated by the dash at the right of Π. The exponential factor is to be repeated with each of the factors $1 - \frac{u}{w}$, and is necessary to the

convergence of the infinite product. For

$$\log \sigma u = \log u + \Sigma' \left\{ \log\left(1 - \frac{u}{w}\right) + \frac{u}{w} + \frac{1}{2}\frac{u^2}{w^2} \right\}$$

$$= \log u - \Sigma' \left\{ \frac{u}{w} + \frac{1}{2}\frac{u^2}{w^2} + \frac{1}{3}\frac{u^3}{w^3} + \ldots - \frac{u}{w} - \frac{1}{2}\frac{u^2}{w^2} \right\};$$

the exponential factor furnishing the last two terms, which are seen at once to be necessary, because $\Sigma \frac{1}{w}$ and $\Sigma \frac{1}{w^2}$ do not converge.

The remaining terms all converge, and the formula becomes

$$\log \sigma u = \log u - \frac{u^3}{3}\Sigma\frac{1}{w^3} - \frac{u^4}{4}\Sigma\frac{1}{w^4} - \ldots$$

The function σu may however be presented in a slightly different form by combining with every pair of values of μ, μ' the corresponding pair with opposite signs, and we have instead of $\Pi\left(1 - \frac{u}{w}\right)$ from $-\infty$ to $+\infty$, $\Pi\left(1 - \frac{u^2}{w^2}\right)$ from 1 to $+\infty$, and the formula becomes

$$\sigma u = u \prod_{1}^{w=+\infty}{}' \left(1 - \frac{u^2}{w^2}\right) e^{\frac{u^2}{w^2}}. \qquad \begin{gathered} w = 2\mu\omega + 2\mu'\omega', \\ \mu,\ \mu' = 1,\ 2 \ldots + \infty. \end{gathered}$$

Developed in an infinite series

$$\sigma u = u - \frac{u^5}{4}\Sigma\frac{1}{w^4} - \frac{u^7}{6}\Sigma\frac{1}{w^6} - \ldots$$

The coefficients in this series are integral functions of two constants,

$$g_2 = 2^2.3.5\Sigma'_w \frac{1}{w^4};\quad g_3 = 2^2.5.7.\Sigma'_w \frac{1}{w^6},$$

called the invariants of the corresponding sigma-function, and which are functions of course of the half periods ω, ω'.

The series for σu then takes the form

$$\sigma u = u + * - \frac{g_2 u^5}{2^4.3.5} - \frac{g_3 u^7}{2^3.3.5.7} - \frac{g_2^2 u^9}{2^9.3^2.5.7} - \frac{g_2 g_3 u^{11}}{2^7.3^2.5^2.7.11} - \ldots$$

The sigma function is not an elliptic function, and does not possess an addition-theorem in the usual sense, neither is it periodic; but on increasing the argument by one period, 2ω, we have

$$\sigma(u + 2\omega) = -e^{2\frac{\sigma'\omega}{\sigma\omega}(u+\omega)}.\sigma u$$

and in like manner

$$\sigma(u + 2\omega') = -e^{2\frac{\sigma'\omega'}{\sigma\omega'}(u+\omega')}.\sigma u$$

Weierstrass writes $p\omega + q\omega' = \tilde{\omega}$, $p\eta + q\eta' = \tilde{\eta}$

$$\frac{\sigma'\omega}{\sigma\omega} = \eta,\ \frac{\sigma'\omega'}{\sigma\omega'} = \eta';$$

so that in general $\qquad \sigma(u+2\tilde{\omega}) = \mp e^{2\tilde{\eta}(u+\tilde{\omega})}\sigma u.$

Although σu does not possess an addition theory, the contrary is the case with the second logarithmic derivative $\frac{\sigma'' u}{\sigma u}$, for which Weierstrass has employed again a modified letter

$$\begin{aligned} \wp u &= -\frac{d^2}{du^2}\log \sigma u \\ &= \frac{1}{u^2} + \Sigma'_w\left(\frac{1}{(u-w)^2} - \frac{1}{w^2}\right) \\ &= \frac{1}{u^2} + * + \frac{g_2}{2^2.5}u^2 + \frac{g_3}{2^2.7}u^4 + \dots \end{aligned}$$

where g_2, g_3 are the invariants of σu before mentioned. It is especially to be noticed that u occurs in only one term with a negative exponent, and that the constant term of the series is null. The function $\wp u$ is an elliptic function of the second degree, and is moreover the simplest possible doubly-periodic function. Among the many interesting relations, we ought to notice that, the half-periods of $\wp u$ being ω and ω', if we write $\omega + \omega' = \omega''$

$$\wp\omega = e_1, \quad \wp\omega'' = e_2, \quad \wp\omega' = e_3,$$

and further

$$(\wp' u)^2 = 4(\wp u - \wp\omega)(\wp u - \wp\omega'')(\wp u - \wp\omega') = 4(\wp u - e_1)(\wp u - e_2)(\wp u - e_3)$$

$= 4\wp^3 u - g_2\wp u - q_3$, whence $e_1 \times e_2 \times e_3 = 0$, $\Sigma e_\lambda e_\mu = -\frac{1}{4}g_2$, $e_1e_2e_3 = \frac{1}{4}q_3$, or, if we write $\wp u = s$, $\wp u$ appears as the elliptic function corresponding to the integral

$$u = \int \frac{ds}{\sqrt{4(s-e_1)(s-e_2)(s-e_3)}},$$

For the sum of two arguments, the function $\wp(u+v)$ is expressible as a rational function of $\wp u$, $\wp v$, $\wp' u$, $\wp' v$, for example,

$$\wp(u \pm v) = \frac{1}{4}\left[\frac{\wp' u \mp \wp' v}{\wp u - \wp v}\right]^2 - \wp u - \wp v.$$

As to the derivatives of $\wp u$; the remarkable fact deserves notice that all the derivatives of an even order are entire functions of $\wp u$ itself. For instance

$$\wp'' u = 6\wp^2 u - \tfrac{1}{2}g_3\,; \; \wp^{\text{iv}} u = 120\wp^3 u - (12g_2 - 18g_2)\wp u - 12g_3.$$

Another remarkable relation between the functions $\wp$ and σ is easily deduced, namely

$$\wp u - \wp v = -\frac{\sigma(u+v)\,\sigma(u-v)}{\sigma^2 u\, \sigma^2 v},$$

which Professor Schwarz is in the habit of calling the "pocket edition" of the elliptic functions. The following proposition is one of great importance in the theory of functions.

If ϕu denotes any elliptic function of the r^{th} degree with the periods 2ω, $2\omega'$, and if σu has the same pair of periods, then we can always determine the $2r+1$ quantities $u_1, u_2, \ldots u_r;\ v_1, v_2, \ldots v_r,\ C$ so that

$$\phi(u) = C.\frac{\sigma(u-u_1)\,\sigma(u-u_2)\ldots\sigma(u-u_r)}{\sigma(u-v_1)\,\sigma(u-v_2)\ldots\sigma(u-v_r)};$$

which proposition is capable of inversion. An analogous theorem in regard to $\wp u$ is, if $u_0, u_1, u_2, \ldots u_n$ denote $n+1$ independent variables, then the function

$$\phi(u_0, u_1, u_2, \ldots u_n) = \begin{vmatrix} 1 & \wp u_0 & \wp' u_0 \ldots \wp^{(n-1)} u_0 \\ 1 & \wp u_1 & \wp' u_1 \ldots \wp^{(n-1)} u_1 \\ 1 & \wp u_2 & \wp' u_2 \ldots \wp^{(n-1)} u_2 \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ 1 & \wp u_n & \wp' u_n \ldots \wp^{(n-1)} u_n \end{vmatrix}$$

is an elliptic function of the degree $n+1$ of any one of the arguments $u_0, u_1 \ldots u_n$. In general "every unique elliptic function $\phi(u)$ is expressible as a rational function of $\wp u$ and the first derivative $\wp' u$ with the same pair of periods 2ω, $2\omega'$ as $\phi(u)$; and in like manner $\wp u$ and $\wp' u$ are expressible as rational functions of ϕu and $\phi' u$".

With the function σu are closely connected the following

$$\sigma_1 u = \frac{e^{-\eta u}\,\sigma(\omega+u)}{\sigma\omega} = \frac{e^{\eta u}\,\sigma(\omega-u)}{\sigma\omega}$$

$$\sigma_2 u = \frac{e^{-\eta'' u}\,\sigma(\omega''+u)}{\sigma\omega''} = \frac{e^{\eta'' u}\,\sigma(\omega''-u)}{\sigma\omega''}$$

$$\sigma_3 u = \frac{e^{-\eta' u}\,\sigma(\omega'+u)}{\sigma\omega'} = \frac{e^{\eta' u}\,\sigma(\omega'-u)}{\sigma\omega'}$$

where ω, ω' are the half periods, and $\omega+\omega'=\omega''$, $\frac{\sigma'\omega}{\sigma\omega}=\eta$, $\frac{\sigma'\omega'}{\sigma\omega'}=\eta'$, $\eta+\eta'=\eta''$. By inserting in the "pocket edition" for v the values respectively ω, ω'', ω', we have

$$\wp u - e_1 = \left(\frac{\sigma_1 u}{\sigma u}\right)^2, \quad \wp u - e_2 = \left(\frac{\sigma_2 u}{\sigma u}\right)^2, \quad \wp u - e_3 = \left(\frac{\sigma_3 u}{\sigma u}\right)^2$$

whereby the following relations are established for the differences of the roots. Remembering that $\wp\omega = e_1$, $\wp\omega'' = e_2$, $\wp\omega' = e_3$.

$$\sqrt{e_1-e_2} = \frac{\sigma_2\omega}{\sigma\omega}, \quad \sqrt{e_2-e_3} = \frac{\sigma_3\omega''}{\sigma\omega''}, \quad \sqrt{e_1-e_3} = \frac{\sigma_3\omega}{\sigma\omega}$$

$$\sqrt{e_2-e_1} = \frac{\sigma_1\omega''}{\sigma\omega''}, \quad \sqrt{e_3-e_2} = \frac{\sigma_2\omega'}{\sigma\omega'}, \quad \sqrt{e_3-e_1} = \frac{\sigma_1\omega'}{\sigma\omega'}$$

where we assume $e_1 > e_2 > e_3$. If now we assume $R\left(\frac{\omega'}{\omega i}\right) > 0$, that is, the real component of the complex $\frac{\omega'}{\omega\sqrt{-1}} > 0$, so that in the geometrical representation

the point ω' lies "above" the right line joining $u=0$ and $u=\omega$, then

$$\sqrt{e_3-e_2}=-i\sqrt{e_2-e_3};\quad \sqrt{e_3-e_1}=-i\sqrt{e_1-e_3};\quad \sqrt{e_2-e_1}=-i\sqrt{e_1-e_2}.$$

If now we denote for convenience by λ, μ, ν the indices 1, 2, 3, and write

$$\frac{\sigma u}{\sigma_\lambda u}=\xi_{o\lambda},\quad \frac{\sigma_\mu u}{\sigma_\nu u}=\xi_{\mu\nu},\quad \frac{\sigma_\lambda u}{\sigma u}=\xi_{\lambda o},\ \text{etc.}$$

remembering that
$$\wp' u=-2\frac{\sigma_\lambda u.\sigma_\mu u.\sigma_\nu u}{\sigma u.\sigma u.\sigma u},$$

we easily obtain

$$\frac{d\xi_{o\lambda}}{du}=\xi_{\mu\lambda}\xi_{\nu\lambda},\quad \frac{d\xi_{\mu\nu}}{du}=-(e_\mu-e_\nu)\,\xi_{\lambda o}\xi_{o\nu},\quad \frac{d\xi_{\lambda o}}{du}=-\xi_{\mu o}\xi_{\nu o},$$

whence
$$\left(\frac{d\xi_{o\lambda}}{du}\right)^2=\left(1-(e_\mu-e_\lambda)\xi_{o\lambda}^2\right)\left(1-(e_\nu-e_\lambda)\xi_{o\lambda}^2\right),$$
$$\left(\frac{d\xi_{\mu\nu}}{du}\right)^2=(1-\xi_{\mu\nu}^2)(e_\mu-e_\lambda+(e_\lambda-e_\nu)\xi_{\mu\nu}^2),$$
$$\left(\frac{d\xi_{\lambda o}}{du}\right)^2=(\xi_{\lambda o}^2+e_\lambda-e_\mu)(\xi_{\lambda o}^2+e_\lambda-e_\nu);$$

and the four functions

$$\frac{\sigma u}{\sigma_\lambda u},\quad \frac{1}{\sqrt{e_\mu-e_\lambda}}\frac{\sigma_\mu u}{\sigma_\nu u},\quad \frac{1}{\sqrt{e_\nu-e_\lambda}}\frac{\sigma_\nu u}{\sigma_\mu u},\quad \frac{1}{\sqrt{e_\mu-e_\lambda}\sqrt{e_\nu-e_\lambda}}\frac{\sigma_\lambda u}{\sigma u}$$

satisfy the same differential equation

$$\left(\frac{d\xi}{du}\right)^2=\left(1-(e_\mu-e_\lambda)\xi_2\right)\left(1-(e_\nu-e_\lambda)\xi^2\right).$$

But the English reader will desire to know in what connection the system of Weierstrass stands to the more widely known systems of Jacobi and Legendre. If we define the k of Jacobi by the equation

$$k^2=\frac{e_2-e_3}{e_1-e_3},$$

then the following relations are established between the sigma-quotients and Jacobi's functions. We give only three as specimens, replacing the λ, μ, ν by 1, 2, 3.

$$\frac{\sigma u}{\sigma_3 u}=\frac{1}{\sqrt{e_1-e_3}}\,\mathrm{sn}\,(\sqrt{e_1-e_3}.u,\ k)$$
$$\frac{\sigma_1 u}{\sigma_3 u}=\mathrm{cn}\,(\sqrt{e_1-e_3}.u,\ k)$$
$$\frac{\sigma_2 u}{\sigma_3 u}=\mathrm{dn}\,(\sqrt{e_1-e_3}.u,\ k).$$

Not all the sigma-quotients are so nearly identical with Jacobi's functions, but in all cases the argument u appears multiplied with the same factor $\sqrt{e_1-e_3}$ which is the largest of the three root-differences.

In the defining equation $k^2=\frac{e_2-e_3}{e_1-e_3}$ and the corresponding one $k'^2=\frac{e_1-e_2}{e_1-e_3}$ both of these quantities if real must be greater than zero and less than unity.

They will be real if the points in the plane representing e_1, e_2, e_3 lie in the same straight line, when mod. e_2 must be intermediate between mod. e_1 and mod. e_3 in magnitude. Then if we understand by K and K' the simplest values of the integrals
$$\int_0^1 \frac{dx}{\sqrt{1-x^2.1-k^2x^2}}; \int_0^1 \frac{dx}{\sqrt{1-x^2.1-k^2x^2}}$$
respectively, taking those values of the radicals whose real components are positive, we shall have
$$\omega_1\sqrt{e_1-e_3}=K,\ \omega_3\sqrt{e_1-e_3}=iK',$$
$$\omega_2=\omega_1+\omega_3,$$
and $2\omega_1$, $2\omega_3$ are the primitive pair of periods for the before mentioned $\wp u$, so that as above
$$\wp\omega_1=e_1,\ \wp\omega_2=e_2,\ \wp\omega_3=e_3.$$

It ought to be mentioned that $\sigma_1 u$, $\sigma_2 u$, $\sigma_3 u$ can also be defined in the same simple manner as σu by means of infinite products. If we write
$$w_1=(2\mu+1)\omega+2\mu'\omega', \qquad w_2=(2\mu+1)\omega+(2\mu'+1)\omega'$$
$$w_3=2\mu\omega+(2\mu'+1)\omega', \qquad [\mu,\mu'=0,\ \pm 1,\ \pm 2\ldots\pm\infty]$$
then in general, for $\lambda=1, 2, 3$,
$$\sigma_\lambda u=e^{-\frac{1}{2}e_\lambda u^2}\Pi_{w_\lambda}\left(1-\frac{u}{w_\lambda}\right)e^{\frac{u}{w_\lambda}+\frac{1}{2}\frac{u^2}{w_\lambda^2}}.$$
Finally to show the relation in which the sigma functions stand to the ϑ-functions of Jacobi, we find
$$\sigma u=\frac{2\omega}{\pi}e^{2\eta\omega v^2}.\frac{2h^{\frac{1}{4}}\sin v\pi-2h^{\frac{9}{4}}\sin 3v\pi+2h^{\frac{25}{4}}\sin 5v\pi-\ldots}{2h^{\frac{1}{4}}-3.2.h^{\frac{9}{4}}+5.2h^{\frac{25}{4}}-\ldots}=2\omega e^{2\eta\omega v^2}.\frac{\vartheta_0(v)}{\vartheta_0'(o)}$$
$$\sigma_1 u=e^{2\eta\omega v^2}.\frac{2h^{\frac{1}{4}}\cos v\pi+2h^{\frac{9}{4}}\cos 3v\pi+2h^{\frac{25}{4}}\cos 5v\pi+\ldots}{2h^{\frac{1}{4}}+2h^{\frac{9}{4}}+2h^{\frac{25}{4}}+\ldots}=e^{2\eta\omega v^2}.\frac{\vartheta_1(v)}{\vartheta_1(o)}$$
$$\sigma_2 u=e^{2\eta\omega v^2}.\frac{1+2h\cos 2v\pi+2h^4\cos 4v\pi+2h^9\cos 6v\pi+\ldots}{1+2h+2h^4+2h^9+\ldots}=e^{2\eta\omega v^2}.\frac{\vartheta_2(v)}{\vartheta_2(o)}$$
$$\sigma_3 u=e^{2\eta\omega v^2}.\frac{1-2h\cos 2v\pi+2h^4\cos 4v\pi-2h^9\cos 6v\pi+\ldots}{1-2h+2h_4-2h^9+\ldots}=e^{2\eta\omega v^2}.\frac{\vartheta_3(v)}{\vartheta_3(o)}$$
where $h=e^{\frac{\omega'}{\omega}\pi i}$, $v=\frac{u}{2\omega}$, $\eta=\frac{\sigma'\omega}{\sigma\omega}$.

The functions $\vartheta_0(v)$, $\vartheta_1(v)$, $\vartheta_2(v)$, $\vartheta_3(v)$ as here employed coincide respectively with Jacobi's $\vartheta_1(xq)$, $\vartheta_2(xq)$, $\vartheta_3(xq)$, $\vartheta(xq)$, if we write $v\pi=x$ and $h=q$.

But anything more than a slight account of Weierstrass' system, showing in particular its main points of contact with Jacobi's, would be beyond the intention of this paper. It is to be hoped that Weierstrass' ideas in the function-theory will soon find that widespread recognition which they undoubtedly merit. In a future paper I hope to exhibit the system in greater detail, in particular the formulæ of transformation, showing their analogies to the formulæ of Jacobi.